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Introduction

- In this topic, we will
  - Describe real-valued functions of two and more variables
  - Describe tangent surfaces in two dimensions
  - Discuss the generalization of Taylor series
  - Discuss simultaneous roots of n expressions in n variables
  - Frame our root-finding problem as one of finding a zero of a vector-valued function of a vector variable

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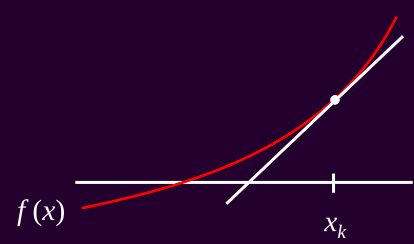
Tangent lines

- Recall Newton's method:
  - Convert a non-linear algebraic expression into a linear problem
  - Find the expression of the tangent line at $(x_k, f(x_k))$
  - The tangent line is a single linear equation in a single unknown
    - It is trivial to find the solution



Root-finding problems

- Recall that we will convert a non-linear algebraic equation into a root-finding problem:

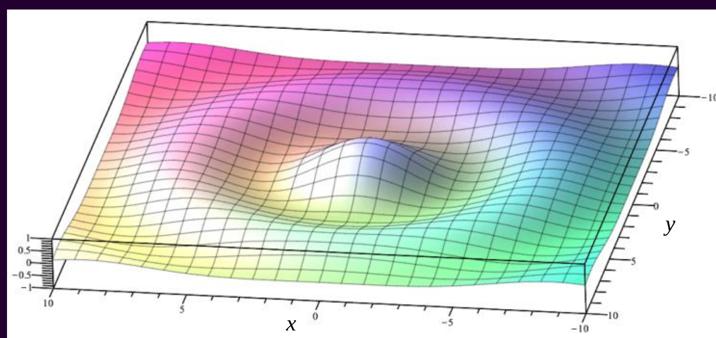
$$x^2 + 2x - 4y + y^2 + 1 = \cos(xy) + 3$$

$$x^2 + 2x - 4y + y^2 - 2 - \cos(xy) = 0$$

Real-valued functions of two real variables

- Suppose we have a real-valued function of two variables

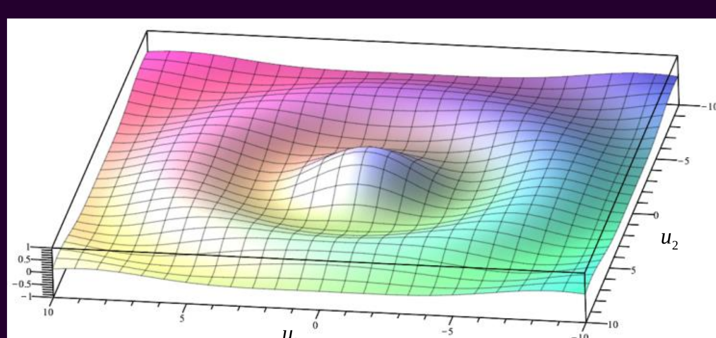
$$f(x, y) = J_0\left(\sqrt{x^2 + y^2}\right)$$



Real-valued functions of a vector variable

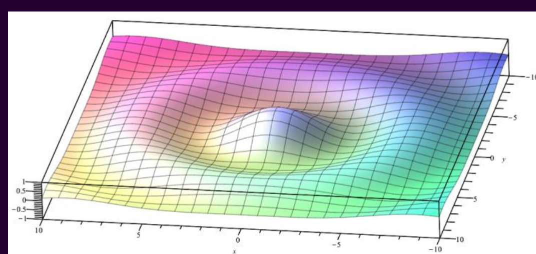
- It is best to think of this as a real-valued function of a vector variable:

$$f(\mathbf{u}) = J_0\left(\sqrt{u_1^2 + u_2^2}\right)$$



Tangent planes in higher dimensions

- Recall that given a differentiable function,
  we can find a tangent line at any point
  - Given a differentiable function of a 2-dimensional vector variable,
    we can find a tangent plane at any point
  - Given a differentiable function of *n*-dimensional vector variable,
    we can find a tangent $(n - 1)$-dimensional *hyperplane* at any point



Tangent planes in higher dimensions

- In one dimension: $f(x) \approx f(x_0) + \frac{d}{dx} f(x_0)(x - x_0)$

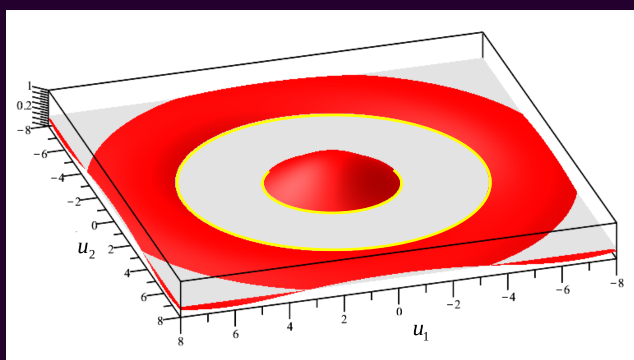
- In two dimensions:
$$f(\mathbf{u}) \approx f(\mathbf{u}_0) + \vec{\nabla} f(\mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) = f(\mathbf{u}_0) + \begin{pmatrix} \frac{\partial}{\partial u_1} f(\mathbf{u}_0) \\ \frac{\partial}{\partial u_2} f(\mathbf{u}_0) \end{pmatrix} \cdot (\mathbf{u} - \mathbf{u}_0)$$

- In *n* dimensions:
$$f(\mathbf{u}) \approx f(\mathbf{u}_0) + \vec{\nabla} f(\mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) = f(\mathbf{u}_0) + \begin{pmatrix} \frac{\partial}{\partial u_1} f(\mathbf{u}_0) \\ \vdots \\ \frac{\partial}{\partial u_n} f(\mathbf{u}_0) \end{pmatrix} \cdot (\mathbf{u} - \mathbf{u}_0)$$

Approximating the solution to a non-linear algebraic equation

Zeros of a function of two variables

- Given a surface, it tends to be zero along curved lines



$$f(\mathbf{u}) = J_0\left(\sqrt{u_1^2 + u_2^2}\right)$$

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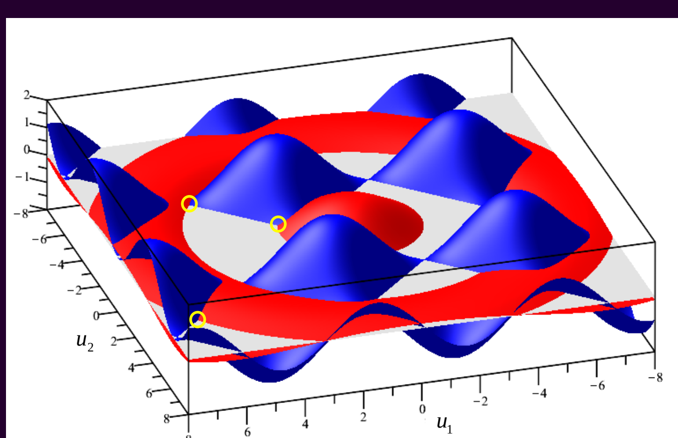
Approximating the solution to a non-linear algebraic equation

A vector-valued function

- Given two surfaces, there are isolated points were both surfaces are simultaneously zero



$$f(\mathbf{u}) = J_0\left(\sqrt{u_1^2 + u_2^2}\right)$$

$$g(\mathbf{u}) = \sin(u_1) - \cos(u_2)$$

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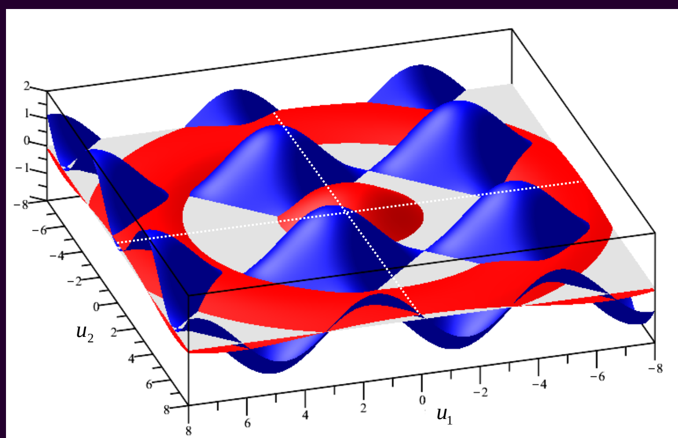
Solutions to equations

- Given two expressions, let us think of them a vector-valued function of a vector variable



$$f_1(\mathbf{u}) = J_0\left(\sqrt{u_1^2 + u_2^2}\right)$$

$$f_2(\mathbf{u}) = \sin(u_1) - \cos(u_2)$$

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} J_0\left(\sqrt{u_1^2 + u_2^2}\right) \\ \sin(u_1) - \cos(u_2) \end{pmatrix}$$

$$\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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Our problem

- Thus, we will have an *n*-dimensional vector-valued function of an *n*-dimensional vector variable $\mathbf{f}(\mathbf{u})$
  - For example,

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 10(u_2 - u_1) \\ u_1(28 - u_3) - u_2 \\ u_1 u_2 - \tfrac{8}{3}u_3 \end{pmatrix}$$

- We want to find values of $\mathbf{u}$ such that $\mathbf{f}(\mathbf{u}) = \mathbf{0}$
  - In this case, we have:

$$\mathbf{u} = \mathbf{0} \qquad \mathbf{u} = \begin{pmatrix} 6\sqrt{2} \\ 6\sqrt{2} \\ 27 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} -6\sqrt{2} \\ -6\sqrt{2} \\ 27 \end{pmatrix}$$

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Fixed-point iteration

- Some of you keeners may have noted the following:

$$\mathbf{0} = \begin{pmatrix} 10(u_2 - u_1) \\ u_1(28 - u_3) - u_2 \\ u_1 u_2 - \frac{8}{3} u_3 \end{pmatrix} \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 10u_2 - 9u_1 \\ u_1(28 - u_3) \\ \frac{3}{8} u_1 u_2 \end{pmatrix}$$

- Thus, solving $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ is the same as solving $\mathbf{g}(\mathbf{u}) = \mathbf{u}$ and so apply fixed-point iteration

$$\mathbf{g}(\mathbf{u}) = \begin{pmatrix} 10u_2 - 9u_1 \\ u_1(28 - u_3) \\ \frac{3}{8} u_1 u_2 \end{pmatrix}$$

  – Thus, start with an initial guess $\mathbf{u}_0$, and then $\mathbf{u}_1 \leftarrow \mathbf{g}(\mathbf{u}_0)$

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Our problem

- In most cases, we will not be able to solve $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ exactly
  – Instead, we will start out with an approximation

$$\mathbf{f}(\mathbf{u}_0) \approx \mathbf{0}$$

  – Then, like with a real-valued function of a real variable,
  we will devise a Newton-like method but for higher dimensions
  – We will find a sequence of vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, ... where hopefully

$$\left\|\mathbf{f}(\mathbf{u}_0)\right\|_2 > \left\|\mathbf{f}(\mathbf{u}_1)\right\|_2 > \left\|\mathbf{f}(\mathbf{u}_2)\right\|_2 > \cdots$$

$$\lim_{k\to\infty} \left|f(x_k)\right| = 0 \qquad \lim_{k\to\infty} \left\|\mathbf{f}(\mathbf{u}_k)\right\|_2 = 0$$

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Summary

- Following this topic, you now
  - Are aware that differentiable function of two variables is smooth and has tangent planes
  - Know that, as with linear equations,
    we require n expressions in n variables
  - Know that we will be expressing this as a vector-valued function of a vector variable
  - Understand the idea of finding the simultaneous root of n expressions in n variables
  - Are aware that we will use a Newton-like method but in higher dimensions

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Approximating the solution to a non-linear algebraic equation

References

[1] https://en.wikipedia.org/wiki/Nonlinear_system
[2] https://en.wikipedia.org/wiki/Bessel_function

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Acknowledgments

Anda Su for noting a mistake in Slide 13 in the conversion of the system of non-linear equations to **x** = *g*(**x**)

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Approximating the solution to a non-linear algebraic equation

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

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